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A CONJUGACY INVARIANT FOR REDUCIBLE SOFIC SHIFTS AND ITS SEMIGROUP CHARACTERIZATIONS

BY

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ABSTRACT

We introduce a notion of magic words and, through them, we present a lattice of sub-synchronizing subshifts which describes the synchronizing parts of a sofic shift S. We show that topological conjugacy maps sub-synchronizing subshifts onto sub-synchronizing subshifts, it preserves their mutual relationship (i.e. the corresponding lattices are isomorphic) and the corresponding covers within the Krieger covers are topologically conjugate. Using the magic words, a full characterization of the syntactic monoid of a shift of finite type is given. We show that a synchronizing deterministic presentation of every sub-synchronizing subshift of S can be seen within a two-sided ideal of the syntactic monoid of S.

Introduction

Sofic shifts appeared in symbolic dynamics with the paper of Weiss [19] where they were defined as images of shifts of finite type under factor maps. Fischer [5] showed that every sofic shift can be presented by a finite directed labeled graph. This implies that the factor language of a sofic shift is a regular language, and thus, it has a finite syntactic semigroup.

Synchronizing words have been studied since the beginning of automata theory. They correspond to the constant functions of the transition monoid of an automaton. Synchronizing words have also been considered in symbolic dynamics and in coding theory, where sometimes they are called "magic" or "resolving blocks" (see for example [12]). In [10] they are called "finitary" blocks. Blanchard and Hansel [2] introduced "coded systems" and "synchronizing systems" which are irreducible but not necessarily sofic, as a generalization of the

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irreducible sofic shifts. Doris and Ulf Fiebig [4] investigated coded synchronizing shifts not necessarily sofic. Sofic shifts with synchronizing presentations (not necessarily irreducible) were investigated in [7].

In this paper, in Section 1, we introduce a notion of "magic synchronizing" words of sofic shifts which, in some sense, captures the synchronizing part of a deterministic presentation of a sofic shift. In fact, magic words determine the set of periodic points of a sofic shift that have singletons as preimages in the Krieger covers (right or left) (Proposition 1.9). Via these words we define sub-synchronizing subshifts of sofic shifts. The sub-synchronizing subshifts of a sofic shift form a lattice. We show that a topological conjugacy maps sub-synchronizing subshifts onto sub-synchronizing subshifts and their mutual relationship is preserved (Proposition 1.12 and Corollary 1.13). The covers of sub-synchronizing subshifts within the Krieger covers of conjugate sofic shifts are also conjugate (Corollary 1.14). This invariant is significant for the reducible sofic shifts, since every irreducible sofic shift is synchronizing. For this purpose we present an example (1.4) of two sofic shifts that have the same entropy and the same zeta function, but one is synchronizing and the other is not.

Several general properties of the syntactic monoid of a sofic shift are given in Section 2 (Corollary 2.5). We show that a sofic shift has a sub-synchronizing subshift iff the ideal formed by the constant words in the syntactic semigroup has a regular D-class. It follows that the regular D-classes in this ideal are preserved by a conjugacy (Corollary 2.7). At the end of Section 2, using magic words and the right Krieger cover, we give a full characterization of the syntactic monoid of a shift of finite type (Proposition 2.8) which is a restatement of the characterization stated in [13].

An intrinsic characterization of a synchronizing sofic shift via the syntactic monoid was given in [7] and we use a similar approach in Section 3 to characterize the sub-synchronizing shifts via certain right ideals. We show that the structure of the sub-synchronizing subshifts and their presentations can be obtained from the syntactic monoid through the regular D-classes of the ideal formed by the constant words (Corollary 3.5).

We end our presentation with a discussion on co-synchronizing and bi-synchronizing sofic shifts.

We start with some definitions and background properties.

Let A be a finite set which we will call an **alphabet**. Finite sequences of elements in A are called **words**. The set of all words (i.e. the free monoid generated by A) is denoted by A^*. The set of all words with positive length (i.e.

the free semigroup generated by A) is denoted by A^+. A subset of A^* is called a **language**.

The **length** of the word $x = a_1 \cdots a_k$ is k and is denoted by $|x|$. The identity element in A^* is the empty word denoted by 1. We will denote by $A^{\geq k}$ the subset of A^* containing the words with length greater than or equal to k.

Let $G = (V, E, \lambda)$ be a finite directed graph with vertices (states) V, edges E and a labeling function λ: $E \to A$. We define two functions: **source** s: $E \to V$ and **target** t: $E \to V$ representing the source and the target of an edge. A **sofic shift** $S = S(G)$ is the set of all bi-infinite sequences obtained by reading the labels of bi-infinite paths of G. We say that G is a **presentation** of S. We will always assume that G is trimmed such that every vertex lies on a bi-infinite path, i.e. the functions s and t are onto.

The **graph shift** S_G presented by the graph G is the sofic shift presented by $(V, E, 1_E)$, i.e. the shift obtained by taking the alphabet to be $A = E$ and the labeling function to be $\lambda = 1_E$, the identity on E.

If the discrete topology is assigned to A and the product topology is assigned to $A^{\mathbb{Z}}$, then the sofic shift S is a compact subset of $A^{\mathbb{Z}}$. The **shift** map σ: $A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ is defined by $(\sigma(\alpha))_i = \alpha_{i+1}$. It is easy to see that σ is a homeomorphism and that S is shift invariant.

The **factor language** $F(S)$ of a sofic shift S is the set of all words which appear in elements of S, i.e. $F(S) = \{a_1 \cdots a_n | \exists \alpha \in S, \exists j \in \mathbb{Z}, \alpha_{j+1} \cdots \alpha_{j+n} = a_1 \cdots a_n\}$. We will call the elements of $F(S)$ **factors**. Because of the closure property of S, the factor language $F(S)$ uniquely determines S, i.e. $S_1 = S_2$ iff $F(S_1) = F(S_2)$. The left infinite sequence $\cdots \alpha_{-2}\alpha_{-1}$ of $\alpha \in S$ will be called **the left ray** of α and denoted by α^-. Sometimes we will abuse this notation and will write α^- for $\cdots \alpha_{-i-2}\alpha_{-i-1}\alpha_{-i}$ for a positive integer i. Similarly the right infinite sequence $\alpha_0\alpha_1 \cdots$ will be called **the right ray** of α and denoted by α^+. We will write $\alpha = \alpha^-\alpha^+$. The set of factors that appear in $\alpha \in S$ will be denoted by $F(\alpha)$. For a language L, $F(L)$ denotes the set of factors of words in L, i.e. $F(L) = \{y | \exists x, z \in A^*, xyz \in L\}$.

The sequence $\alpha \in S$ is **periodic** if there is $u = a_1 \cdots a_k \in F(S)$ such that for all $i \in \mathbb{Z}$, $\alpha_{ki+1} \ldots \alpha_{ki+k} = u$. In that case we will write $\alpha = u^\omega$.

A presentation G of a sofic shift S may be regarded as a finite state automaton with all states being initial and terminal, which recognizes the factor language $F(S)$. The language recognized by a presentation G will be denoted by $L(G)$. Thus the factor language $F(S)$ is always regular. It is easily seen that $F(S)$ is also **prolongable** (i.e. $\forall x \in F(S), \exists a, b \in A$ such that $axb \in F(S)$) and **factorial** (meaning each factor of a word in $F(S)$ is also in $F(S)$). The converse is also true: if a language L is factorial, prolongable and regular (**FPR**), then there is a sofic shift S such that $L = F(S)$ ([2] or [7]).

A presentation G is **deterministic** if for every vertex in G the outgoing edges are labeled distinctly. This presentation is also known as right resolving. It is very well known that every sofic shift has a deterministic presentation. In a deterministic presentation, for an edge e with $s(e) = v$, $t(e) = w$ and $\lambda(e) = a$ we will write $va = w$. We extend this notation for paths by writing $vx = w$ for a path from v to w with a label x. The **right context** of v in G is the set $R(v, G) = \{x \in A^* | \exists w \in V, vx = w\}$. We will write $R(v)$ when G is understood. Similarly we define the **left context** of v to be the set of words which are labels of all paths that end at v. We say that G is **reduced** if no two vertices in G have same right contexts. The process of identifying vertices with same right contexts is called **reduction.** It is easy to see that if G' is obtained from G by reduction, then G' is reduced and the reduction induces a label preserving graph homomorphism from G onto G' and both G and G' present the same sofic shift.

For a word $x \in L \subset A^*$ we define the **right context of** x **in** L to be the set $R(x, L) = \{y \in A^* | xy \in L\}$. Again, we will write $R(x)$ when L is understood from the context. Similarly, we define the **left context of** x to be the set of words which can precede x in L.

A vertex v is **synchronizing** if there is a word m_v such that every path in G with label m_v ends at v. We say that m_v is synchronizing for G at the vertex v. A presentation G is **synchronizing** if every vertex in G is synchronizing. A sofic shift is called **synchronizing** if it has a synchronizing presentation. It was shown in [7] that every synchronizing sofic shift has a synchronizing deterministic presentation.

Let $L \subseteq A^*$ be a language. The word m is **constant for** L if for all $x_1, x_2 \in A^*$ if $x_1 m$ and $m x_2$ are in L, then $x_1 m x_2$ is also in L. It is clear that if m is a constant then for every x_1 and x_2, $x_1 m$ and $m x_2$ are also constants. A constant word m is called **synchronizing for the language** L if $m \in L$. Note that every synchronizing word of a presentation G of a sofic shift S is also a constant word for the language $F(S)$ ([18]). Following this note and Proposition 0.1 below, we see that for every sofic shift S, the factor language $F(S)$ has a synchronizing word.

PROPOSITION 0.1 ([5] and [10]): *If G is a reduced deterministic presentation of a sofic shift S, then G has a synchronizing word.*

Often we will refer to the synchronizing words for $F(S)$ as synchronizing words for S.

PROPOSITION 0.2 ([7]): *S is synchronizing iff for every $x \in F(S)$ there is a synchronizing word m for $F(S)$ such that $mx \in F(S)$.*

We say that $G' = (V', E', \lambda')$ is a **subgraph** of the sofic presentation $G = (V, E, \lambda)$ if $V' \subseteq V$, every edge $e \in E$ with $s(e), t(e) \in V'$ is also in E' and $\lambda' = \lambda | E'$. G' is **a terminal subgraph** of G if $\forall e \in E \; s(e) \in V' \Rightarrow t(e) \in V'$. The subgraph of G determined by the set of synchronizing vertices will be denoted by $T(G)$. Clearly, $T(G)$ is a terminal subgraph of G, and in the case when G is synchronizing, $T(G) = G$.

Let $\mathcal{K}$ be a presentation of S obtained by constructing the usual minimal deterministic automaton (with unique initial state) which recognizes the factor language $F(S)$ and erasing the vertices which do not lie on bi-infinite paths. This presentation is called the **right Krieger cover** for S (see [10]). The following proposition says that $T(\mathcal{K})$ is the 'maximal' synchronizing part that a deterministic presentation of S can have.

PROPOSITION 0.3: (a) ([7]) *If G is a reduced deterministic presentation of S, then $T(G)$ is isomorphic to a terminal subgraph of $T(\mathcal{K})$.*

(b) ([7] and [18]) *The set of synchronizing words for S is equal to the set of synchronizing words for $\mathcal{K}$.*

(c) *For each word $x \in F(S)$ there is $y \in F(S)$ such that xy is synchronizing for S.*

Proof: The proof of (c) follows the same line of argument as Lemma 2 in [5]. ∎

An FPR language L is called **transitive** if for all $x, y \in L$ there is a z such that $xzy \in L$. A sofic shift is called **irreducible** if $F(S)$ is transitive.

PROPOSITION 0.4 ([5]): *A sofic shift is irreducible iff it can be presented with a strongly connected graph.*

It is well known that irreducible sofic shifts, together with subshifts of finite type and non-wondering sofic shifts, are synchronizing (see for example [7]).

The strongly connected components of a presentation G will be called **irreducible components**. It is clear that if a vertex of a strongly connected component is in $T(G)$, then the whole strongly connected component is in $T(G)$.

An irreducible component C of G is called **initial** if there is no path from any other irreducible component to a vertex in C.

We mention some well known facts and definitions about continuous shift commuting maps on sofic shifts (see for example [12], originally in [6]). Let $S_1, S_2 \subseteq A^{\mathbb{Z}}$ be two sofic shifts. If S_1 and S_2 are shifts over different alphabets, B_1 and B_2, we will assume that $A = B_1 \cup B_2$. A function $f\colon S_1 \to S_2$ is called a k-**block map** if there is an integer $k \geq 0$ and a function $f_*\colon A^{2k+1} \to A$ such that $(f(\alpha))_i = f_*(\alpha_{i-k}, \ldots, \alpha_i, \ldots, \alpha_{i+k})$. It can be shown that every continuous shift commuting map is a k-block map for some k ([6] or [12]). A **topological conjugacy** is a bijective k-block map. It follows that an inverse of a k-block map is a k'-block map for some k'.

NOTE 0.5: *Every k-block map f is also an m-block map f' for every $m \geq k$ by taking $f'_*(\alpha_0, \ldots, \alpha_{2m}) = f_*(\alpha_{m-k}, \ldots, \alpha_{m+k})$.*

A **subshift of finite type** (SFT) is defined to be a sofic shift S with the property that there is $n > 0$ such that every $x \in F(S)$ with length $\geq n$ is synchronizing. The graph shift S_G of a graph G is an SFT with $n = 1$, since every edge is a synchronizing word for S_G. The labeling function $\lambda\colon E \to A$ of G generates a 0-block map Λ from $E^{\mathbb{Z}}$ to $A^{\mathbb{Z}}$ and so S is a factor image of S_G, i.e. $\Lambda(S_G) = S$. It is a well known fact that every subshift of finite type is conjugate to a graph shift. Lemma 1.2 in [9] shows that when S is an SFT and $\mathcal{K}$ is its right Krieger cover, then Λ is conjugacy. This implies that in this case, $T(\mathcal{K}) = \mathcal{K}$.

1. Sub-synchronizing subshifts

Let $L \subseteq A^*$ be a language. A synchronizing word m is **magic** for L if there is $x \in L$ such that $mxm \in L$. We will call the magic synchronizing words for $F(S)$ magic synchronizing for the sofic shift S. The set of magic synchronizing words for S will be denoted by $\mathcal{M}(S)$.

A synchronizing word m for a presentation G is called **magic** if there is a path with label m in G within one irreducible component of G. A synchronizing word m of G is **transient** if there are two vertices v_1 and v_2 not in the same irreducible component and $v_1 m = v_2$. Note that a synchronizing word can be both magic and transient for G.

PROPOSITION 1.1: *A word m is a magic synchronizing word for S iff it is a magic synchronizing word for the right Krieger cover $\mathcal{K}$.*

Proof: If m is magic for S, then m is synchronizing for $\mathcal{K}$ (by Proposition 0.3(b)). Let v be the synchronizing vertex for m. There is x such that $mxm \in F(S)$ and so $vxm = v$ and the path starting at v with label xm is within the irreducible component of v. Thus m is magic for $\mathcal{K}$.

Converse: If m is magic for $\mathcal{K}$ then there are vertices v_1 and v_2 within same irreducible component with $v_1 m = v_2$. But there is a path from v_2 to v_1 in $\mathcal{K}$. If x is the label of that path, $mxm \in F(S)$. ∎

NOTE 1.2: *By Proposition* 1.1, *if m is magic for a deterministic presentation G of S then it is magic for $\mathcal{K}$ and for S.*

We say that the shift $S' \subset S$ is a **sub-synchronizing subshift of S** if there is a set of magic synchronizing words $M_{S'}$, for S such that $F(S')$ is the set of factors of the words in the right context of words in $M_{S'}$, i.e. $F(S') = \{y | \exists m \in M_{S'}, \exists x \in A^*, mxy \in F(S)\}$. By definition $F(S')$ is factorial language. $F(S')$ is also prolongable: Let $y \in F(S')$ and $mxy \in F(S)$ for some x. Then there is $a \in A$ such that $mxya \in F(S)$ and so $ya \in F(S')$. Proposition 1.3 below shows that $F(S')$ is also regular. So S' is a sofic shift. We will say that $M_{S'}$ **defines** S'. If M is a set of magic synchronizing words that defines sub-synchronizing subshift S', we will write $S' = \mathcal{S}(M)$. In the case when $M = \{m\}$, we will write $\mathcal{S}(m)$ instead of $\mathcal{S}(\{m\})$.

It is clear that if $m \in F(S')$ is synchronizing for S, then it will be synchronizing for S'. But our definition requires even more. Since every magic word is in the set of factors of its right context, i.e. $m \in F(R(m))$, the set $M_{S'}$ is contained in $F(S')$. So S' inherits a set of magic words from S.

Obviously, S has a sub-synchronizing subshift iff $\mathcal{M}(S) \neq \emptyset$, i.e. iff the right Krieger cover has a magic synchronizing word.

PROPOSITION 1.3: *Let S be a sofic shift and $\mathcal{K}$ its right Krieger cover. If S' is a sub-synchronizing subshift of S then there is a terminal subgraph of $T(\mathcal{K})$ which is a presentation of S'.*

Proof: By Proposition 1.1, m is a magic word for S iff m is magic for $\mathcal{K}$. Let S' be defined by the set $M_{S'}$, i.e. $\mathcal{S}(M_{S'}) = S'$, and let V_M be the set of vertices of $\mathcal{K}$ which contain words from $M_{S'}$ in their left contexts. Let $V' = \{v | \exists w \in V_M, \exists x, wx = v\}$. Obviously V' is not empty and V' determines a terminal subgraph G' of $T(\mathcal{K})$. It is also clear that $M_{S'} \subseteq L(G')$. We need to show that G' is a presentation of S'. By definition of G' and S', $L(G') \subset F(S')$. Let $x \in F(S')$. By the definition of S', there is a word $m \in M_{S'}$ such that $myx \in F(S)$ for some y. If $v_m \in V_M$ is a synchronizing vertex in $\mathcal{K}$ corresponding to m, then there is a path in G' starting at v_m with label yx. Hence, x is in the language recognized by G', i.e. $L(G') = F(S')$. ∎

COROLLARY 1.4: *S has only a finite number of sub-synchronizing subshifts.*

Proof: Follows from the fact that $\mathcal{K}$ has only a finite number of terminal subgraphs. ∎

We note here that if S is an irreducible sofic shift, then S is the only sub-synchronizing subshift of S.

PROPOSITION 1.5: *A sub-synchronizing shift is synchronizing.*

Proof: Let S' be sub-synchronizing of S and $M_{S'}$ a set of magic words such that $\mathcal{S}(M_{S'}) = S'$. By the definition of magic words, each word in $M_{S'}$ is a factor of its right context, so $M_{S'} \subset F(S')$. We observe that $m \in M_{S'}$ is also synchronizing for S'. Let x and y be such that $xm, my \in F(S')$. By definition of S', there is $m' \in M_{S'}$ and z such that $m'zxm \in F(S)$. Since m is synchronizing for $F(S)$, $m'zxmy \in F(S)$ and thus $xmy \in F(S')$. Now by Proposition 0.2 and the definition of S', S' is synchronizing. ∎

We note here that it follows from the definition of a synchronizing sofic shift and the above proposition that

COROLLARY 1.6: *S is synchronizing iff it is a sub-synchronizing subshift of itself.*

COROLLARY 1.7: *If S' is sub-synchronizing of S then there is a finite set $M_{S'}$ of magic words that defines S'.*

Proof: Let G' be the terminal subgraph of $T(\mathcal{K})$ which is a presentation of S'. By the definition of G' (in the proof of Proposition 1.3) each initial irreducible component of G' has a magic word. Let $M_{S'}$ be the set of magic synchronizing words such that $x \in M_{S'}$ iff x is in an initial component of G' and if $x \neq y$ then x and y belong to different initial components of G'. Then $M_{S'}$ is a finite set of magic words which defines S'. ∎

If we compare Proposition 0.2 and Corollaries 1.6 and 1.7 we see that the factor language of a synchronizing sofic shift is contained in the set of factors of the words in the right context of a finite set of magic words. In other words, S is synchronizing iff there is a finite set of magic words M such that $F(S) = \bigcup_{m\in M} F(R(m))$.

PROPOSITION 1.8: *Let G be a reduced deterministic presentation of S and let $T(G)$ be a presentation of a subshift S'. If S' is not sub-synchronizing, then G has a transient synchronizing word which is not magic in S.*

Proof: Since $T(G)$ is a presentation of a sofic shift, the initial irreducible components are presentations of irreducible sofic shifts. Assume that every initial component of $T(G)$ has a path whose label is a magic word in G. Let $M_{S'}$ be the set of the magic words which are labels of paths in an initial component of $T(G)$. Then S' is a sub-synchronizing of S defined via $M_{S'}$. So, if $T(G)$ is not a presentation of a sub-synchronizing subshift then there is an initial component C of $T(G)$ which 'does not contain' a magic word for G. But each vertex in C is synchronizing for G, so the synchronizing words for vertices in C must be transient and not magic for G. ∎

With the following proposition we show what is the relationship between the periodic points of S and the magic words. It shows that the magic words determine the set of periodic points in S whose preimages in the right Krieger cover are singletons.

PROPOSITION 1.9: *Let $\mathcal{K} = (V, E, \lambda)$ be the right Krieger cover for S and let $\Lambda\colon S_{\mathcal{K}} \to S$ be the 0-block map generated by λ. Let $\alpha \in S$ be a periodic sequence. Then $|\Lambda^{-1}(\alpha)| = 1$ iff there is a magic synchronizing word m such that $\alpha = m^\omega$.*

Proof: Let $Y = \{\alpha \in S | \forall i \in \mathbb{Z}, \exists j < i, \alpha_j\alpha_{j+1}\cdots\alpha_i$ is synchronizing for $S\}$. By Lemma 2.6 in [10], for each $\alpha \in Y$, $|\Lambda^{-1}(\alpha)| = 1$. If $\alpha \in S$ is such that $\alpha = m^\omega$, and m is synchronizing, then $\alpha \in Y$.

Conversely, assume that $\alpha = x^\omega$ and that $|\Lambda^{-1}(\alpha)| = 1$. It is sufficient to show that for some k, x^k is a synchronizing word for $\mathcal{K}$ and thus for S. Suppose that for each k, x^k is not synchronizing for $\mathcal{K}$. Then, for each k, there are at least two paths, $p_1^{(k)}$ and $p_2^{(k)}$, in $\mathcal{K}$ with $\lambda(p_1^{(k)}) = \lambda(p_2^{(k)}) = x^k$. Since $\mathcal{K}$ is deterministic, we can consider the end vertices of these paths, u_k and v_k, respectively. There are finite number of vertices in $\mathcal{K}$, so there are a finite number of pairs (u_k, v_k), i.e. for some $k' > k$ the paths $p_i^{(k)}$ and $p_i^{(k')}$ have the same initial and terminal vertices $(i = 1, 2)$. So $(u_k, v_k) = (u_{k'}, v_{k'})$. If $u_k \neq v_k$ for all k, then $|\Lambda^{-1}((x^k)^\omega)| \geq 2$ (i.e. there are two distinct loops with label x^s for some s) which is a contradiction to the assumptions. Thus $v_k = u_k$ for some k, i.e. x^k is a synchronizing word for $\mathcal{K}$. ∎

Let m_1 and m_2 be magic synchronizing words for S. We say that m_1 and m_2 are **equivalent,** and write $m_1 \sim m_2$, if there are words x, y such that $m_1xm_2, m_2ym_1 \in F(S)$. It follows from the definition of magic synchronizing words and the definition of $\sim$ that $\sim$ is an equivalence relation. We will donote the set of equivalence classes of $\mathcal{M}(S)$ by $\mathcal{E}$ and the equivalence class of m by $\mathbf{m}$. It follows from Corollaries 1.4 and 1.7 and the definition of $\sim$ that $\mathcal{E}$ is finite.

PROPOSITION 1.10: *For every $m \in \mathbf{m}$, $\mathcal{S}(\mathbf{m}) = \mathcal{S}(m)$.*

Proof: It is clear that $\mathcal{S}(m) \subseteq \mathcal{S}(\mathbf{m})$. Let $z \in F(\mathcal{S}(\mathbf{m}))$. Then there is $m' \in \mathbf{m}$ and x such that $m'xz \in F(S)$. Since $m \sim m'$, there are y_1 and y_2 such that

$m'y_1m, my_2m' \in F(S)$. But m' is synchronizing, so $my_2m'xz \in F(S)$, i.e. $z \in F(\mathcal{S}(m))$. ∎

We define a partial order on $\mathcal{E}$ with $\mathbf{m}_1 \leq \mathbf{m}_2$ if there are $m_1 \in \mathbf{m}_1, m_2 \in \mathbf{m}_2$ and x such that $m_2xm_1 \in F(S)$. It is easy to check that the relation $\leq$ is a partial order on $\mathcal{E}$ since each of $m \in \mathbf{m}$ are synchronizing and magic. Similarly, as in Proposition 1.10, we can show that

$$\mathbf{m}_1 \leq \mathbf{m}_2 \quad \text{implies} \quad \mathcal{S}(\mathbf{m}_1) \subseteq \mathcal{S}(\mathbf{m}_2)$$

which defines a lattice of sub-synchronizing subshifts of S. Elements of this lattice are the sub-synchronizing subshifts of S and the $\emptyset$. The operations of this lattice are defined by $S' \vee S'' = S' \cup S''$ and $S' \wedge S'' = S' \cap S''$ if $S' \cap S''$ is a sub-synchronizing subshift of S and $S' \wedge S'' = \emptyset$ otherwise. We will denote this lattice by $\mathcal{L}$.

We have that $\mathcal{S}(\mathbf{m}_1) \vee \mathcal{S}(\mathbf{m}_2) = \mathcal{S}(\mathbf{m}_1) \cup \mathcal{S}(\mathbf{m}_2) = \mathcal{S}(\mathbf{m}_1 \cup \mathbf{m}_2)$. Let S' be a sub-synchronizing subshift of S defined through a set of magic words M, i.e. $S' = \mathcal{S}(M)$. Let $\mathbf{M} = \{\mathbf{m} | \mathbf{m} \cap M \neq \emptyset\}$. Then by Proposition 1.10,

$$S' = \mathcal{S}(M) = \mathcal{S}(\mathbf{M}) = \bigcup_{\mathbf{m} \in \mathbf{M}} \mathcal{S}(\mathbf{m}).$$

So every element in $\mathcal{L}$ is defined by a union of elements in $\mathcal{E}$.

In fact, the partial order relation preserves the order of the strongly connected components of the right Krieger cover. If m_1 is equivalent to m_2 then there are paths within one irreducible component of $\mathcal{K}$ with labels m_1 and m_2. We can look at each equivalence class as a representative of an irreducible component of $\mathcal{K}$ containing a magic synchronizing word. And if $\mathbf{m}_1 \leq \mathbf{m}_2$, then there is a path from the irreducible component of $\mathbf{m}_2$ to the irreducible component of $\mathbf{m}_1$.

LEMMA 1.11: *For all $k > 0$ and every $\mathbf{m} \in \mathcal{E}$ there is $m \in \mathbf{m}$ with $|m| > k$ and $m^i \in \mathbf{m}$ for every $i = 1, 2, 3, \ldots$.*

Proof: Let m be a magic synchronizing word for S and $\mathbf{m}$ its equivalence class in $\mathcal{E}$. Then there is x such that $mxm \in F(S)$. Since m is synchronizing, $(mx)^i \in F(S)$ is magic synchronizing for each $i > 0$ and $(mx)^i \in \mathbf{m}$. Let j be such that $|(mx)^j| > k$. Then $m' = (mx)^j$ is in $\mathbf{m}$ and $(m')^i \in \mathbf{m}$ for every $i = 1, 2, 3, \ldots$. ∎

We note that if m satisfies the condition of Lemma 1.11, then m^j, $j = 1, 2, 3, \ldots$ satisfies the condition of Lemma 1.11 too.

In order to facilitate the proofs of the main results of this section we will need the following definition.

Let $f\colon S_1 \to S_2$ be a k-block map defined via $f_*\colon A^{2k+1} \to A$. We define $\phi_f\colon A^{\geq(2k+1)} \to A^*$ in the following way: if $s \geq 2k+1$ then

$$\phi_f(a_1 \cdots a_s) = f_*(a_1, \ldots, a_{2k+1}) \cdots f_*(a_{s-2k}, \ldots, a_s).$$

By the above definition, if a word x has length $s \geq 2k+1$ then the word $\phi_f(x)$ has length $s-2k$. Also, it follows directly that if $x \in F(\alpha)$ for some $\alpha \in S_1$ then $\phi_f(x) \in F(f(\alpha))$.

PROPOSITION 1.12: *Let $S_1, S_2 \subseteq A^{\mathbb{Z}}$ be two sofic shifts and let $f\colon S_1 \to S_2$ be a topological conjugacy. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the lattices of sub-synchronizing subshifts of S_1 and S_2, respectively. Then there is an isomorphism $\Phi_f\colon \mathcal{L}_1 \to \mathcal{L}_2$.*

Proof: Since f is topological conjugacy, f is a k_1-block map, and there is an inverse map f^{-1}. It is well known that the inverse of a topological conjugacy is also a topological conjugacy (see for example [6] or [12]), i.e. f^{-1} is a k_2-block map for some k_2. Let $k' = \max\{k_1, k_2\}$. By Note 0.5, both f and f^{-1} are k'-block maps.

Let $\mathcal{E}_1$, $\mathcal{E}_2$ be the sets of equivalence classes of $\mathcal{M}(S_1)$, $\mathcal{M}(S_2)$ respectively and let $\mathbf{m} \in \mathcal{E}_1$. Let $\hat{n}$ be a magic synchronizing word in $\mathbf{m}$ such that $|\hat{n}| \geq k'$ and $\hat{n}^i \in \mathbf{m}$ for all $i = 1, 2, 3, \ldots$ (exists by Lemma 1.11). Since $\mathcal{E}_1$ is finite, it follows from Lemma 1.11 that there is $k > k'$ such that for all $\mathbf{m} \in \mathcal{E}_1$, there is $\hat{m} \in \mathbf{m}$ with $|\hat{m}| = k$ and $\hat{m}^i \in \mathbf{m}$ for all $i = 1, 2, 3, \ldots$. (For example, for each $\mathbf{m}$ we pick $\hat{n} \in \mathbf{m}$ with $|\hat{n}| > k'$ and take k to be the least common multiplier of the lengths of $\hat{n}$. So $\hat{m} = \hat{n}^s$ for some s.) By Note 0.5, both f and f^{-1} are k-block maps.

Define $\Phi_f\colon \mathcal{L}_1 \to \mathcal{L}_2$ with the following:

$$\forall \mathbf{m} \in \mathcal{E}_1, \Phi_f(\mathcal{S}(\mathbf{m})) = \Phi_f(\mathcal{S}(\hat{m})) = \mathcal{S}(\phi_f(\hat{m}^6))$$

and $\Phi_f(\mathcal{S}(\hat{m}_1 \cup \hat{m}_2)) = \Phi_f(\mathcal{S}(\hat{m}_1)) \cup \Phi_f(\mathcal{S}(\hat{m}_2))$.

First we will show that $\phi_f(\hat{m}^6) = m' \in F(S_2)$ is a magic synchronizing word for S_2. The length of $\hat{m}^6$ is $6k$, and so the length of m' is $4k$. Let $\alpha = \alpha^- \hat{m}^6 \alpha^+$ such that $\alpha_{-3k} \cdots \alpha_0 \cdots \alpha_{3k-1} = \hat{m}^6$ and let $\beta = f(\alpha)$. Then $\beta_{-2k} \cdots \beta_0 \cdots \beta_{2k-1} = \phi_f(\hat{m}^6) = m'$. Since $f^{-1}(\beta) = \alpha$, we have that

$$(\star) \qquad \phi_{f^{-1}}(m') = \phi_{f^{-1}}(\beta_{-2k} \cdots \beta_0 \cdots \beta_{2k-1}) = \alpha_{-k} \cdots \alpha_0 \cdots \alpha_{k-1} = \hat{m}\hat{m}.$$

Let $y_1, y_2 \in F(S_2)$ be such that $y_1 m'$ and $m' y_2$ are in $F(S_2)$. There are β' and β'' in S_2 such that $\beta' = \beta_1^- y_1 m' \beta_1^+$ and $\beta'' = \beta_2^- m' y_2 \beta_2^+$. Since S_2 is shift invariant, we can take $\beta'_{-2k} \cdots \beta'_0 \cdots \beta'_{2k-1} = \beta''_{-2k} \cdots \beta''_0 \cdots \beta''_{2k-1} = m'$. Let $\alpha' = f^{-1}(\beta') = \alpha_1^- \hat{m}\hat{m}\alpha_1^+$ and $\alpha'' = f^{-1}(\beta'') = \alpha_2^- \hat{m}\hat{m}\alpha_2^+$. Since $\hat{m}$ is synchronizing and S_1 is compact, $\bar{\alpha} = \alpha_1^- \hat{m}\hat{m}\alpha_2^+ \in S_1$ and $f(\bar{\alpha}) = \gamma \in S_2$. We will show that $y_1 m' y_2 \in F(\gamma)$. Since S_1 is shift invariant, we can assume that

$$\bar{\alpha}_{-k} \cdots \bar{\alpha}_0 \cdots \bar{\alpha}_{k-1} = \hat{m}\hat{m} = \alpha'_{-k} \cdots \alpha'_0 \cdots \alpha'_{k-1} = \alpha''_{-k} \cdots \alpha''_0 \cdots \alpha''_{k-1}.$$

In that case, for each $i > 0$ we have:

$$\begin{aligned} \gamma_{-i} &= f_*(\bar{\alpha}_{-i-k}, \ldots, \bar{\alpha}_{-i}, \ldots, \bar{\alpha}_{-i+k}) = f_*(\alpha'_{-i-k}, \ldots, \alpha'_{-i}, \ldots, \alpha'_{-i+k}) = \beta'_{-i}, \\ \gamma_i &= f_*(\bar{\alpha}_{i-k}, \ldots, \bar{\alpha}_i, \ldots, \bar{\alpha}_{i+k}) = f_*(\alpha''_{i-k}, \ldots, \alpha''_i, \ldots, \alpha''_{i+k}) = \beta''_i, \end{aligned}$$

and for $i = 0$ we have $\gamma_0 = \phi_f(\hat{m}\hat{m}\bar{\alpha}_k) = \phi_f(\hat{m}\hat{m}\alpha''_k) = \beta''_0 = \beta'_0$. So $\gamma = (\beta')^-(\beta'')^+ = \beta_1^- y_1 \beta'_{-2k} \cdots \beta'_{-1} \beta''_0 \cdots \beta''_{2k-1} y_2 \beta_2^+ = \beta_1^- y_1 m' y_2 \beta_2^+$ which shows that $y_1 m' y_2$ is in the factor set $F(S_2)$, i.e. m' is synchronizing for $F(S_2)$. We see that m' is magic, since $\hat{m}^{12} \in F(S_1)$ and $\phi_f(\hat{m}^{10}) = m'm' \in F(S_2)$.

Now we show that Φ_f is well defined. Let $\hat{n} \in \mathbf{m}$ with $|\hat{n}| \geq k'$ and $\hat{n}^i \in \mathbf{m}$ for all $i > 0$. Let $\phi_f(\hat{n}^6) = n'$. Since $\hat{m}, \hat{n} \in \mathbf{m}$, there are x, y such that $\hat{n}x\hat{m}, \hat{m}y\hat{n} \in F(S_1)$. But $\hat{m}$ and $\hat{n}$ are magic synchronizing words and satisfy Lemma 1.11, so $\hat{n}^i x \hat{m}^j y \hat{n}^l \in F(S_1)$ for all $i, j, l > 0$. Hence $\phi_f(\hat{n}^6 x \hat{m}^6 y \hat{n}^6) = n'x'm'y'n'$, i.e. $n' \in \mathbf{m}'$.

In a symmetric way, we can define $\Phi_{f^{-1}}: \mathcal{L}_2 \to \mathcal{L}_1$ which would be a well defined function too. We will show that $\Phi_{f^{-1}} = \Phi_f^{-1}$ which will prove that Φ_f is a one-to-one correspondence. Let $\Phi_{f^{-1}} \circ \Phi_f(\mathcal{S}(\mathbf{m})) = \mathcal{S}(\mathbf{n})$. Let $\phi_f(\hat{m}^6) = m'$ and $\phi_{f^{-1}}((n')^6) = n$ for $m' \sim n'$. Since n' is a magic synchronizing word for S_2, we have that $m'x(n')^6 y m' \in F(S_2)$ for some x, y. Then by $(\star)$ we have that $\phi_{f^{-1}}(m'x(n')^6 y m') = \hat{m}\hat{m}z_1 n z_2 \hat{m}\hat{m}$ for some z_1 and z_1. So $\hat{m}, n \in \mathbf{m}$, i.e. $\mathbf{m} = \mathbf{n}$. By a symmetric argument we have that $\Phi_f \circ \Phi_{f^{-1}}(\mathcal{S}(\mathbf{m}')) = \mathcal{S}(\mathbf{m}')$.

We end the proof by showing that Φ_f preserves the operations in $\mathcal{L}_1$. By the definition, Φ_f preserves $\vee$. Let $\mathbf{m} \leq \mathbf{n} \in \mathcal{E}_1$, i.e. $\mathcal{S}(\mathbf{m}) \subseteq \mathcal{S}(\mathbf{n})$. Consider $\hat{n} \in \mathbf{n}$ and $\hat{m} \in \mathbf{m}$ that satisfy Lemma 1.11. Then there is x such that $\hat{n}x\hat{m} \in F(S_1)$ and so $\hat{n}^i x \hat{m}^i \in F(S_1)$ for all $i > 0$. Let $\Phi_f(\mathcal{S}(\mathbf{n})) = \mathcal{S}(\mathbf{n}')$ and $\Phi_f(\mathcal{S}(\mathbf{m})) = \mathcal{S}(\mathbf{m}')$ such that $\phi_f(\hat{n}^6) = n'$, $\phi_f(\hat{m}^6) = m'$. Then $\phi_f(\hat{n}^6 x \hat{m}^6) = n'zm' \in F(S_2)$ and so $\mathbf{m}' \leq \mathbf{n}'$. Now the proof follows from the fact that if $\mathcal{S}(\mathbf{m}_1) \cap \mathcal{S}(\mathbf{m}_2) = \mathcal{S}(\mathbf{n})$ then $\mathbf{n} \leq \mathbf{m}_1$ and $\mathbf{n} \leq \mathbf{m}_2$. ∎

COROLLARY 1.13: *Let S_1 and S_2 be sofic shifts, $f: S_1 \to S_2$ be a topological conjugacy and $\Phi_f: \mathcal{L}_1 \to \mathcal{L}_2$ be the isomorphism from Proposition 1.12. For every sub-synchronizing subshift S', $\Phi_f(S')$ is topologically conjugate to S', i.e. $f|S' = \Phi_f(S')$.*

Proof: Assume that $S' = \mathcal{S}(\mathbf{m})$ and $S'' = f|S'$. Then $\Phi_f(S') = \mathcal{S}(\mathbf{m}')$ for some $\mathbf{m}' \in \mathcal{E}_2$. Let $y \in F(S'')$. We will use a similar argument as in Proposition 1.12. If y is the empty word, $m'y = m' \in F(\mathcal{S}(\mathbf{m}'))$. So we can assume that y has a positive length, i.e. $|y| = r+1$ for some $r \geq 0$. Let $\beta \in S''$ be such that $\beta = \beta_1^- y \beta_1^+$ with $y = \beta_0 \cdots \beta_r$. Let $\alpha = f^{-1}(\beta)$, i.e. $f(\alpha) = \beta$ and let $z = \alpha_{-k} \cdots \alpha_0 \cdots \alpha_{r+k}$. Then there is $x \in F(S')$ such that $\hat{m}xz \in F(S_1)$, where $\hat{m}$ satisfies Lemma 1.11 and $|\hat{m}| = k$. Hence $\hat{m}^6 xz \in F(S')$ and $\phi_f(\hat{m}^6 xz) = m'y'y$. So $y \in F(\mathcal{S}(\mathbf{m}'))$, i.e. $S'' \subseteq \mathcal{S}(\mathbf{m}') = \Phi_f(S')$.

Conversely, assume that $y \in \Phi_f(S') = \mathcal{S}(\mathbf{m}')$. Then there is $x \in F(S_2)$ such that $m'xy \in F(S_2)$. Let $\beta = \beta^- m'xy\beta^+$ be in $\mathcal{S}(\mathbf{m}')$ such that $\beta^- = ((m')^\omega)^-$. Then it follows from $(\star)$ that $f^{-1}(\beta) = \alpha^-\alpha^+$ is such that $\alpha^- = (\hat{m}^\omega)^-$, i.e. $\alpha \in \mathcal{S}(\mathbf{m}) = S'$ and so $\beta \in f(S')$ and $y \in F(S'')$.

The general case follows from the fact that the Φ_f preserves $\vee$. ∎

Krieger in [10] showed that if S_1 is a sofic shift conjugate to S_2, then this conjugacy lifts to a conjugacy between the graph shifts of the right Krieger covers $\mathcal{K}(S_1)$ and $\mathcal{K}(S_2)$. With the following corollary we see that a similar statement is true for sub-synchronizing subshifts and their synchronizing deterministic presentations.

Let S_1 and S_2 be two synchronizing sofic shifts. Let $\mathcal{K}_1 = (V_1, E_1, \lambda_1)$ and $\mathcal{K}_2 = (V_2, E_2, \lambda_2)$ be their right Krieger covers, respectively.

COROLLARY 1.14: *Let $f: S_1 \to S_2$ be a topological conjugacy, and S_1', S_2' be sub-synchronizing subshifts of S_1, S_2 respectively. If $f|S_1' = S_2'$, then there is a conjugacy $\varphi: S_{G_1} \to S_{G_2}$ between the graph shifts S_{G_1}, S_{G_2}, where G_1, G_2 are the subgraphs of $\mathcal{K}_1, \mathcal{K}_2$ that are presentations of S_1', S_2' respectively.*

Proof: Let $f: S_1 \to S_2$ be a topological conjugacy. Let $\Lambda_1: S_{\mathcal{K}_1} \to S_1$ and $\Lambda_2: S_{\mathcal{K}_2} \to S_2$ be the 0-block maps determined by the labeling functions λ_1 and λ_2, respectively. By Theorem 2.13 in [10] we have that there is a conjugacy $\phi: S_{\mathcal{K}_1} \to S_{\mathcal{K}_2}$ such that the following diagram commutes:

$$\begin{array}{ccccccc}
S_{G_1} & \xrightarrow{i_1} & S_{\mathcal{K}_1} & \xrightarrow{\phi} & S_{\mathcal{K}_2} & \xleftarrow{i_2} & S_{G_2} \\
\downarrow{\scriptstyle \Lambda_1|S_{G_1}} & & \downarrow{\scriptstyle \Lambda_1} & & \downarrow{\scriptstyle \Lambda_2} & & \downarrow{\scriptstyle \Lambda_2|S_{G_2}} \\
S_1' & \xrightarrow{j_1} & S_1 & \xrightarrow{f} & S_2 & \xleftarrow{j_2} & S_2'
\end{array}$$

where i_1, i_2 and j_1, j_2 are the inclusion maps.

We will show that $\varphi = \phi | S_{G_1}$ is a conjugacy from S_{G_1} to S_{G_2}. By Corollary 1.7 and Proposition 1.10, there is a finite set of magic words M in S_1 such that $S_1' = \mathcal{S}(M)$. Using Lemma 1.11 we can choose that all $m \in M$ have the same length ($|m| = k$) and, as in the proof of Proposition 1.12, we can consider f as a k-block map. Let $\pi \in S_{G_1}$ and $\alpha = \Lambda_1(\pi)$. Since S_{G_1} is a graph shift of G_1, by Propositions 0.2, 1.5, 1.10 and definitions of magic words, for each $i > 0$ there are $\pi^{(i)} \in S_{G_1}$ such that

$$\pi^{(i)} = \cdots \pi^{(i)}_{-i-1} \pi^{(i)}_{-i} \pi_{-i+1} \pi_{-i+2} \cdots$$

and $\Lambda_1(\pi^{(i)}) = \cdots ((m_i)^\omega)^- y_i \alpha_{-i+1} \alpha_{-i+2} \cdots$, where m_i is a magic synchronizing word in M. Since $|\Lambda_1^{-1}((m_i)^\omega)| = 1$ (Proposition 1.9), $\Lambda_1^{-1}(\Lambda_1(\pi^{(i)})) = \pi^{(i)}$. The sequence $\{\pi^{(i)}\}_{i=1}^\infty$ converges to π, and since Λ_1 is continuous, $\{\Lambda_1(\pi^{(i)})\}_{i=1}^\infty$ converges to α. Then $\{f(\Lambda_1(\pi^{(i)}))\}_{i=1}^\infty$ converges to $f(\alpha) = \beta$. The diagram above commutes, so $\{\phi(\pi^{(i)})\}_{i=1}^\infty$ converges to $\phi(\pi)$ and $\Lambda_2(\phi(\pi)) = \beta$.

Let $\beta^{(i)} = f(\Lambda_1(\pi^{(i)}))$. Then $\beta^{(i)} = f(\Lambda_1(\pi^{(i)})) = \cdots ((m_i')^\omega)^- y_i' \beta_{-i+k+1} \beta_{-i+k+2} \ldots$, are in S_2'. Since $|\Lambda_2^{-1}(\beta^{(i)})| = 1$ and the sequence $\{\beta^{(i)}\}_{i=1}^\infty$ converges to β, we have that $\Lambda_2^{-1}(\beta^{(i)}) = \phi(\pi^{(i)})$ is in S_{G_2} for each $i > 0$. But S_{G_2} is closed, so the limit of $\{\phi(\pi^{(i)})\}_{i=1}^\infty$ is in S_{G_2}, i.e. $\phi(\pi) \in S_{G_2}$.

By reversing the argument, $\phi^{-1}(\bar{\pi}) \in S_{G_1}$ for each $\bar{\pi}$ in S_{G_2}. ∎

From Corollary 1.14 it follows that for a given sofic shift S, we can consider a lattice of covers of sub-synchronizing subshifts that is preserved by a conjugacy. The elements of this lattice (also preserved by a conjugacy) are graph shifts corresponding to the presentations of sub-synchronizing subshifts within the right Krieger cover of S.

We note here that the proof of Corollary 1.14 also shows that if S is synchronizing, then the set of sequences $Y = \{\beta \in S | \ \forall i \in \mathbb{Z}, \exists j < i, \alpha_j \alpha_{j+1} \cdots \alpha_i$ is synchronizing for $S\}$ is dense in S. This is a generalization of Lemma 2.5 in [10] which proves the same statement for irreducible sofic shifts.

Examples: The purpose of the first three examples is to show different appearances (or non-appearances) of magic and transient synchronizing words and sub-synchronizing shifts. The fourth example shows two sofic shifts. The first one is synchronizing and the second one isn't, but both have the same zeta function and the same entropy. There are several conjugacy invariants, and most would distinguish non-conjugate sofic shifts by distinguishing their irreducible components. This example shows that the concept of synchronizing is a conjugacy invariant for reducible sofic shifts which can distinguish non-conjugate sofic shifts when some other invariants may fail.

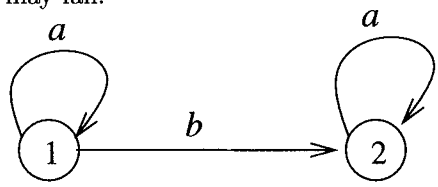

Figure 1.

1.1. Let $G = (V, E, \lambda)$ be defined by

$$V = \{1, 2\}, \quad E = \{(1, a, 1), (1, b, 2), (2, a, 2)\}$$

where the edges are given with (source, label, target) (see Figure 1). The only synchronizing words of the sofic shift S presented with G are words of the form $a^k b a^l$ for $k, l \geq 0$. The reader can verify that this presentation is the right Krieger cover for this shift and thus there is no magic synchronizing word for S. This sofic shift does not have any sub-synchronizing subshifts.

1.2. Let G be given with $V = \{1, 2, 3\}$ and the edges are given with $E = \{(1, a, 1), (1, b, 2), (2, a, 2), (2, b, 3), (3, b, 2)\}$ (see Figure 2). The sofic shift presented with this graph has a synchronizing word ba. In this presentation ba is both transient and magic. The sub-synchronizing shift defined by $M_{S'} = \{ba\}$ is presented with the subgraph containing the vertices $\{2, 3\}$ and is the so-called even shift. The reader can verify that this is the only sub-synchronizing subshift of S.

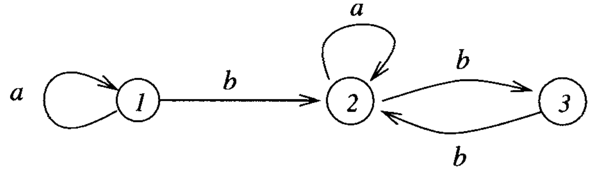

Figure 2.

1.3. Let S be a sofic shift presented with the following graph. The set of vertices is $V = \{1, 2, 3, 4, 5, 6\}$ and the set of edges is $E = \{(1, c, 2), (1, b, 3), (2, b, 1), (2, a, 5), (3, a, 3), (3, b, 4), (4, b, 3), (6, c, 5), (5, a, 6), (5, c, 5)\}$ (see Figure 3 (a)). The reader can easily verify that this presentation is reduced, deterministic and synchronizing. So S is a synchronizing sofic shift and so it is a sub-synchronizing subshift of itself defined via the set $M_S = \{bc\}$. There are only three other sub-synchronizing subshifts of S. One can be defined by $M_{S_1} = \{ab\}$, the other by $M_{S_2} = \{ac\}$ and the third by $M_{S_3} = \{ab, ac\}$. The shift S_1 is presented with the terminal subgraph containing the vertices $\{3,4\}$, S_2 with the vertices $\{5,6\}$ and S_3 with the vertices $\{3,4,5,6\}$. It can be easily seen that there are no other sub-synchronizing subshifts of S. The lattice of sub-synchronizing subshifts of S is presented in Figure 3 (b).

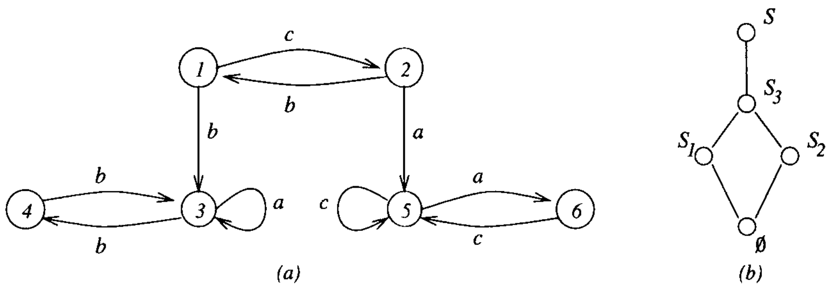

Figure 3.

1.4. Let S_1 and S_2 be presented with G_1 and G_2 as in Figure 4 (a) and (b), respectively. Clearly, G_1 and G_2 have same irreducible components, just 'reversed'. It can be easily seen that the presentation (a) of S_1 is synchronizing. The presentation (b) is contained in the right Krieger cover $\mathcal{K}_2$ of S_2 and the reader can verify that $T(\mathcal{K}_2)$ is the terminal irreducible component of the presentation (b). So, S_2 is not synchronizing. The word cc is synchronizing for both S_1 and S_2, but $M = \{cc\}$ in the first case defines S_1 and in the second case defines a proper subshift of S_2.

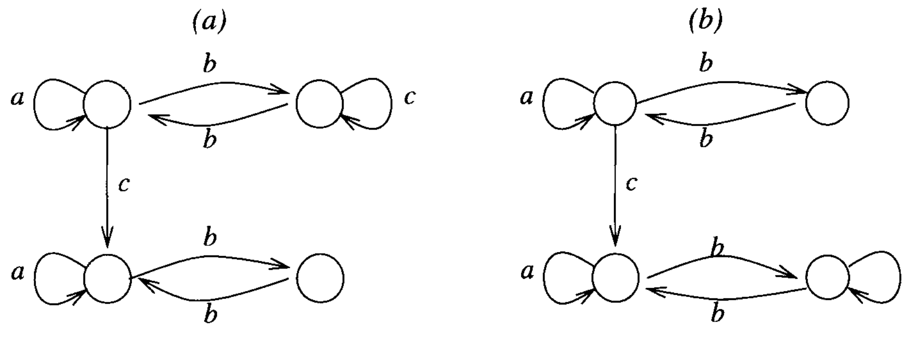

Figure 4.

The **entropy** of a sofic shift S is defined to be the asymptotic growth rate of the number of words in $F(S)$ by

$$h(S) = \lim_{n\to\infty} \frac{1}{n} \log |F_n(S)| \quad \text{where } F_n(S) = A^n \cap F(S).$$

It is well known that the entropy of a graph shift S_G is the logarithm of the spectral radius of the adjacency matrix of G. It is also well known that for a deterministic presentation G of a sofic shift S, the entropies of S_G and S coincide (see for example [1] or [12]). Using this fact it is easy to see that both shifts S_1 and S_2 have entropies $\log 2$.

The zeta function of a sofic shift S is defined by

$$\zeta(t) = \exp\left(\sum_{n=1}^{\infty} \frac{P_n(\sigma)}{n} t^n\right)$$

where $P_n(\sigma)$ is the number of periodic sequences with period n for the shift σ in S. It is known that this function is always rational for sofic shifts and it can be calculated using adjacency matrices and permutations of vertices (see for example [15], also [1], or [12]). In this case, for shifts S_1 and S_2 one can calculate* that

$$\zeta_t(S_1) = \zeta_t(S_2) = \frac{(1+t-3t^2-4t^3+3t^5+t^6)(-t+1)}{(t^2-3t+1+2t^3)(1-2t^2+t^4)}.$$

2. The syntactic monoid of $F(S)$

In this section we give some general properties of the syntactic monoid of the factor language $F(S)$ which follow from the fact that $F(S)$ contains a synchronizing word. We refer the reader to [11] or [14] for the background on finite semigroups and the Green's relations.

A syntactic congruence $\sim_L$ for a language $L \subseteq A^*$ is defined as follows:

$$\forall x, y \in A^*, x \sim_L y \text{ iff } \forall z_1, z_2 \in A^*, z_1 x z_2 \in L \Leftrightarrow z_1 y z_2 \in L.$$

The **syntactic monoid** $\mathrm{Syn}(L)$ of L is the monoid $A^*/_{\sim_L}$. For $x \in A^*$, the element of $\mathrm{Syn}(L)$ with representative x will be written as $[x]$. The operation in $\mathrm{Syn}(L)$ is defined by $[x][y] = [xy]$. It is a well known fact (see [11] or [14]) that L is regular iff $\mathrm{Syn}(L)$ is finite.

Throughout this section we assume that the sofic shift S is not the full shift, i.e. $F(S) \neq A^*$. When $F(S) = A^*$, the syntactic monoid $\mathrm{Syn}(A^*)$ has only one element and the full shift $A^{\mathbb{Z}}$ has only itself as a sub-synchronizing. Note also that if the empty word 1 is a constant for $F(S)$, then $F(S) = A^*$ and $S = A^{\mathbb{Z}}$. So with our assumption, we also have that 1 is not a constant of $F(S)$.

Proposition 2.1 ([2] and [7]): *$L \subset A^*$ is an FPR language iff $\mathrm{Syn}(L)$ has the following properties:*

* This was calculated with aid of Maple V.3.

(i) Syn(L) *is finite.*
(ii) Syn(L) *has a 0 such that* $[x] = 0$ *iff* $x \notin L$.
(iii) *For all non-zero elements* $[x] \in$ Syn(L) *there are* y_1, y_2, *neither equal to 1 such that* $[y_1 x y_2] \neq 0$.

We will denote the syntactic monoid Syn($F(S)$) shortly by Syn. Since it is a finite monoid, the Green's relations J and D define the same partition on Syn and we will interchange D and J in our notation. If X is any of the Green's relations R, L, H, D or J, we will use the notation X_y for the X-class of y. We will write $X_y \leq X_z$ if $y \leq_X z$.

Constants in factorial and prolongable languages, not necessarily regular, are considered in [8]. Propositions 2.2 – 2.4 below hold in the more general case when $F(S)$ is not regular, i.e. S is not necessarily sofic.

PROPOSITION 2.2 ([8]): *If* m *is a synchronizing word for* S *then* $D_{[m]}$ *is* H*-trivial.*

PROPOSITION 2.3 ([8]): *Let* m *be synchronizing for* $F(S)$. *Then* m *is magic for* S *iff* $D_{[m]}$ *is a regular* D*-class.*

PROPOSITION 2.4 ([8]): *Let* m *be synchronizing for* S. *If* $0 \neq r \in$ Syn *is such that* $D_r < D_{[m]}$, *then* r *has a synchronizing word as a representative and* D_r *is not regular.*

Now we can conclude:

COROLLARY 2.5: *Let* $\mathcal{I}_c = \{[x] | x$ *is a constant for* $F(S)\}$. *Then*
(a) $\mathcal{I}_c$ *is a group-free ideal.*
(b) $\mathcal{I}_c$ *has no non-zero idempotents iff* S *has no magic synchronizing words.*
(c) *A non-zero* D*-class in* $\mathcal{I}_c$ *is regular only if it is maximal.*
(d) *Every 0-minimal ideal of* Syn *is a subset of* $\mathcal{I}_c$.

Proof: (a) $\mathcal{I}_c$ is an ideal since every extension of a synchronizing word in $F(S)$ is again synchronizing. $\mathcal{I}_c$ is group-free by Proposition 2.2.

(b) follows from Proposition 2.3.

(c) follows from Proposition 2.4.

(d) Let I be an ideal in Syn. Let $x \in F(S)$ be such that $[x] = t$. By Proposition 0.3 (c) there is $y \in F(S)$ such that xy is synchronizing for S. We take $m = xy$ and $[m] = [xy] \leq_R [x] = t$. So if $t \in I$ then $[m] \in I$. This implies that $\mathcal{I}_c \cap I \neq 0$. So if I is a 0-minimal ideal, then $I = I \cap \mathcal{I}_c$. ∎

The above corollary is a generalization of Proposition 3.3 in [2] to the non-irreducible case. There, it was shown that if S is irreducible (not necessarily sofic), then $\mathcal{I}_c$ is the unique 0-minimal ideal of Syn and it is H-trivial.

With the following two propositions we show that the regular D-classes in $\mathcal{I}_c$ are preserved with topological conjugacy.

PROPOSITION 2.6: *Let m_1, m_2 be two magic synchronizing words for the sofic shift S. Then $\mathbf{m}_1 = \mathbf{m}_2$ iff $D_{[m_1]} = D_{[m_2]}$.*

Proof: Assume that $\mathbf{m}_1 = \mathbf{m}_2$, i.e. there are x_1, x_2 such that $m_1x_1m_2, m_2x_2m_1 \in F(S)$. Then $m_1x_1m_2x_2m_1 \in F(S)$. For every z_1, z_2 such that $z_1m_1z_2 \in F(S)$, since m_1 is synchronizing we have that $z_1m_1x_1m_2x_2m_1z_1 \in F(S)$. So $[m_1x_1m_2x_2m_1] = [m_1]$, or $D_{[m_1]} \leq D_{[m_2]}$. By symmetry, $D_{[m_1]} = D_{[m_2]}$.

Conversely, assume that $D_{[m_1]} = D_{[m_2]}$. By Proposition 2.3, D is regular. So there are $[z_1], [z_2] \in$ Syn such that $[m_1z_1]R[m_1]$, $[m_2z_2]R[m_2]$ and $[m_1z_1], [m_2z_2]$ are idempotents. Then $[m_1z_1]D[m_2z_2]$ and there are x_1, x_2 and y_1, y_2 such that $[x_1m_1z_1x_2] = [m_2z_2]$ and $[y_1m_2z_2y_2] = [m_1z_1]$. But then $[m_2z_2m_2z_2] = [x_1m_1z_1x_2m_2z_2] = [m_2z_2]$. So, $m_1z_1x_2m_2 \in F(S)$. Symmetrically $m_2z_2y_2m_1 \in F(S)$ and hence $\mathbf{m}_1 = \mathbf{m}_2$. ∎

Let S_1, S_2 be two sofic shifts and $\mathcal{I}_c^1$ and $\mathcal{I}_c^2$ be the ideals corresponding to the constant words of the syntactic monoids of S_1, S_2 respectively.

COROLLARY 2.7: *If S_1 is topologically conjugate to S_2, then there is a one-to-one correspondence between the reular D-classes of $\mathcal{I}_c^1$ and the regular D-classes of $\mathcal{I}_c^2$.*

Proof: It follows from Section 1 that there is a 1-1 correspondence between the equivalence classes of magic words of S_1 and the magic words of S_2. From Proposition 2.6 it follows that there is a 1-1 correspondence between the equivalence classes of magic words of a sofic shift and the regualr D classes in the ideal $\mathcal{I}_c$. ∎

We end this section with the following characterization of the subshifts of finite type. Let $\mathcal{I}_c = \{[x] | x \text{ is a constant for } F(S)\}$.

PROPOSITION 2.8: *The sofic shift $S \neq A^{\mathbb{Z}}$ is a shift of finite type if and only if $[1] = \{1\}$ and the set of idempotents $E = \{e | e^2 = e, e \neq 1, e \neq 0\}$ is a non-empty subset of the ideal $\mathcal{I}_c$.*

Proof: Let $S \neq A^{\mathbb{Z}}$ be a SFT and let n be such that every word in $F(S)$ with length $\geq n$ is synchronizing. Let $x \in F(S)$ be such that $x \in [1]$. Since $[1][1] = [1]$, we have that $[x][x] = [x] = [1]$, i.e. $[x^k] = [1]$ for any $k \geq 0$. If $|x| \geq 0$, i.e. if $x \neq 1$, then x^n and so the empty word 1 are synchronizing words for $F(S)$. Thus $F(S) = A^*$ which is a contradiction to our assumption that $S \neq A^{\mathbb{Z}}$. So, $x = 1$.

Now we first note that $\mathrm{Syn}(S)$ has an idempotent $e \neq 1$ and $e \neq 0$. This follows from Corollary 2.5 (b) since every SFT is synchronizing and thus contains a magic synchronizing word.

Let $e \neq 1$ be an idempotent in $\mathrm{Syn}(S)$ with representative $x \in F(S)$. Then $|x| > 0$ and so x^n is a synchronizing word for S. But $[x] = [x][x] = [x^n] = e$, so $e \in \mathcal{I}_c$.

Converse: Suppose that S is a sofic shift such that $[1] = \{1\}$ in $\mathrm{Syn}(S)$ and the set $E = \{e | e^2 = e, e \neq 1, e \neq 0\}$ is a non-empty subset of the ideal $\mathcal{I}_c$. Note that $[1] = \{1\}$ implies that for every $x \in F(S)$, $x \notin H_{[1]}$. Let α be a periodic sequence in S, i.e. $\alpha = y^\omega$. Then for all $i \geq 0$, $y^i \in F(S)$. Since Syn is finite, $[y^k]$ is an idempotent for some k. By the proof of Proposition 2.3, and Corollary 2.5, $[y^k] \in \mathcal{I}_c$ is magic synchronizing. Let $\mathcal{K}$ be the right Krieger cover of S. By Proposition 1.9, $|\Lambda^{-1}(\alpha)| = 1$, so we must have that $k = 1$ and y is a label of a unique cycle in $\mathcal{K}$. Moreover, y must be a primitive word, i.e. y is not of the form z^s for some z and s. So every cycle in $\mathcal{K}$ is uniquely determined by its label.

Let n be the number of vertices in $\mathcal{K}$. We show that every word in $F(S)$ with length $n^2 + 1$ is synchronizing for $\mathcal{K}$. Let $x \in F(S)$ be such that $|x| \geq n^2 + 1$. Assume that x is not synchronizing for $\mathcal{K}$. Then there are two paths $p = e_0 \cdots e_{n^2}$ and $p' = e'_0 \cdots e'_{n^2}$ with label $\lambda(p) = \lambda(p') = x$ and targets $t(p) = t(e_{n^2}) \neq t(p') = t(e'_{n^2})$. By the determinism of $\mathcal{K}$, if $e_i = e'_i$ then $e_j = e'_j$ for all $i \leq j \leq n^2$. Since p and p' have different targets, $e_i \neq e'_i$ for all $0 \leq i \leq n^2$. We consider the pairs of vertices (v_i, v'_i) where $v_i = s(e_i)$ and $v'_i = s(e'_i)$. Since there are only n vertices, there are i and j $(i < j)$ such that $(v_i, v'_i) = (v_j, v'_j)$. But this implies that $y = \lambda(e_i \cdots e_{j-1}) = \lambda(e'_i \cdots e_{j-1})$ is a label of two distinct cycles in $\mathcal{K}$ which is a contradiction to the fact that cycles in $\mathcal{K}$ are uniquely determined by their labels. Thus x must be a synchronizing word for $\mathcal{K}$ and, by Proposition 0.3 (b), x is synchronizing for S. ∎

It is known that a sofic shift is a shift of finite type iff $F(S)$ is a strictly locally testable language (see [16]). A characterization of the syntactic semigroup of a strictly locally testable language is given in [13]. This characterization is equivalent to Proposition 2.8. Here, we give a different proof, using the properties of $\mathcal{K}$ and magic synchronizing words.

Example: It is clear that strictly locally testable languages are group-free (see [16], [17],) and so the syntactic semigroup of a factor set of a shift of finite type has only trivial subgroups. But there are strictly sofic shifts (not of finite type) with syntactic semigroups having only trivial subgroups. The shift presented in Figure

1 is not a shift of finite type, but the syntactic semigroup of its factor language has only three elements: $\{0, [a] = 1, [b]\}$, and it has only trivial subgroups. The ideal $\mathcal{I}_c = \{0, [b]\}$ has no idempotents other than 0. So the non-emptiness condition of the set E in Proposition 2.8 is necessary. In this example we also have that $[1] \neq \{1\}$.

3. Sub-synchronizing subshifts and syntactic monoids

In this section we will characterize the sub-synchronizing subshifts of a given sofic shift S by certain right ideals of the syntactic monoid $\mathrm{Syn}(F(S))$ (or shortly just Syn). We assume that the sofic shift S is not the full shift, i.e. $F(S) \neq A^*$. When $F(S) = A^*$, the syntactic monoid Syn has only one element and the full shift $A^{\mathbb{Z}}$ has only itself as a sub-synchronizing.

We use a similar approach as in [7]. Let S' be a sub-synchronizing subshift of S and let $G = (V, E, \lambda)$ be a reduced deterministic presentation of S such that a terminal subgraph T' of $T(\mathcal{K})$ is a presentation of S'. Such T' exists by Proposition 1.3. Let $C_i = (V_i, E_i, \lambda_i)$ $(i = 1, \ldots, k)$ be the initial irreducible components of T'. Define $\bar{V}_i$ to be the set of all vertices in G connected to V_i, i.e.

$$\bar{V}_i = \{v \in V | \exists v_i \in V_i, \exists x \in L, v_i x = v\}.$$

We fix $v_i \in V_i$ and a magic synchronizing word m_i for v_i. Such m_i exists by the proof of Proposition 1.3.

Define $\Phi_i\colon \bar{V}_i \to \mathrm{Syn}(L)$ with $\forall w \in \bar{V}_i$

$$\Phi_i(w) = [m_i x] \quad \text{iff } v_i x = w.$$

Let $I_i = \Phi_i(\bar{V}_i) \cup \{0\}$.

PROPOSITION 3.1: *For each i, j,*

(a) *Φ_i is a well defined one-to-one function,*
(b) *I_i is a right ideal and*
(c) *$w_1, w_2 \in \bar{V}_i$ belong to the same irreducible component of T' iff $\Phi_i(w_1)$ and $\Phi_i(w_2)$ belong to the same R-class in I_i.*
(d) *If $i \neq j$ then $I_i \cap I_j = 0$.*

Proof: The proof of all (a)–(d) follows the lines of the arguments in Section 8 of [7]. ∎

The previous proposition actually says that there is a one-to-one correspondence between the R-classes of the right ideal I_i and the irreducible components determined by $\bar{V}_i$, and that is a one-to-one correspondence between the elements of the R-classes and the vertices of the corresponding irreducible component of T'.

The R-classes (and the D-classes that they belong to) of I_i that correspond to the initial irreducible components C_i will be called **initial**. Note that the initial R-classes (D-classes) may also be terminal.

PROPOSITION 3.2: *For $[x],[y] \in I_i$, $[x] \leq_R [y]$ iff there is a path from $\Phi_i^{-1}([y])$ to $\Phi_i^{-1}([x])$ in G.*

Proof: $\Rightarrow$ Let $[x] \leq_R [y]$. Then there is a z such that $[x] = [yz]$. Since $[x],[y] \in I_i$, there are x' and y' such that $[x] = [m_i x']$ and $[y] = [m_i y']$. Then $[m_i x'] = [m_i y' z]$. By the definition of Φ_i, there is a path from $\Phi_i^{-1}([m_i y'])$ to $\Phi_i^{-1}([m_i x']) = \Phi_i^{-1}([m_i y' z])$ in G.

$\Leftarrow$ Let $v = \Phi_i^{-1}([x])$ and $w = \Phi_i^{-1}([y])$. Let z be the label of the path from w to v in G. Then by the definition of Φ, $[x] = [yz]$, and thus $[x] \leq_R [y]$. ∎

The previous proposition says that the initial R-classes are maximal R-classes.

Let $I_r = \bigcup_{i=1}^{k} I_i$. Then I_r is a right ideal (as a union of right ideals). It can be shown (see [7]) that for all $[x],[y] \in I_r$, if $[xy] \neq 0$ then $[xy] = [y]$. Note that an R-class is initial in I_r iff it is a maximal R-class in I_r. By Corollary 2.5 (c), an R-class in I_r is initial iff it is regular.

From the synchronizing deterministic presentation T' we obtained a right ideal I_r. Now consider the converse; let I_r be a right ideal of Syn. We define a graph $G(I_r)$ such that the set of vertices of $G(I_r)$ is $I_r - \{0\}$, the set of edges is $\{([x], a, [xa]) \mid [x],[xa] \in I_r - \{0\}, a \in A\}$ and the labeling is defined by $\lambda([x], a, [xa]) = a$. The source of the edge $e = ([x], a, [xa])$ is $s(e) = [x]$ and the target is $t(e) = [xa]$. It is obvious that $G(I_r)$ is deterministic and that different R-classes in I_r correspond to different irreducible components of $G(I_r)$. The graph $G(I_r)$ is not necessarily reduced. Let $G^r(I_r)$ be the reduced graph of $G(I_r)$.

PROPOSITION 3.3: *If $I_r = \bigcup_{i=1}^{k} I_i$, then $G(I_r)$ is a presentation of the sub-synchronizing subshift S' of S and $G^r(I_r) = T'$.*

Proof: Let $x \in F(S')$, i.e. let w_1 and w_2 be two vertices in T' such that $w_1 x = w_2$. Then there is an initial irreducible component C_i in T' such that there is a path from v_i to w_1. Let $v_i y = w_1$. Then $m_i y x \in F(S)$ and so $[m_i y], [m_i y x] \in I_r - \{0\}$. So in $G(I_r)$, $[m_i y]x = [m_i y x]$, i.e. $F(S') \subseteq L(G(I_r))$.

Now let $x \in L(G(I_r))$, i.e. there are x_1 and x_2 such that $[x_1],[x_2] \in I_r - \{0\}$ and $[x_1]x = [x_2]$. Since each I_i is a right ideal, there is an i such that $[x_1],[x_2] \in I_i$

and $[x_2] \leq_R [x_1]$. By Proposition 3.2 there is a path from $\Phi_i^{-1}([x_1])$ to $\Phi_i^{-1}([x_2])$ and so $x \in F(S')$.

Let $e = [m]$ be an idempotent in I_r. Then $[m]$ is in an initial component of $G(I_r)$. By definition of I_r, there is i such that $[m_i x] = [m]$. Assume that $[y] \in I_r$ is such that $ym \in F(S)$. Then $[y]m = [ym] = [ym_i x]$ in $G(I_r)$. By definition of Φ_i, $\Phi_i^{-1}([m]) = \Phi_i^{-1}([m_i x]) = \Phi_i^{-1}([ym_i x]) = \Phi_i^{-1}([ym])$. This implies $[m] = [ym]$ and $[m]$ is synchronizing for $G(I_r)$. So $G(I_r)$ is synchronizing. Thus $G^r(I_r)$ is synchronizing, reduced and deterministic. By Proposition 0.3 (a), $G^r(I_r) = T'$. ∎

PROPOSITION 3.4: *The sofic shift S has a sub-synchronizing subshift S' iff the syntactic monoid* Syn *of S has a right ideal I_r satisfying the following two properties:*

(i) *An R-class of I_r is regular iff it is maximal in I_r.*
(ii) *For all $s, t \in$ Syn and an idempotent $e \in I_r$, if $s \leq_L e$ and $t \leq_R e$ then $0 \neq st \leq_J e$.*

And in that case, $G(I_r)$ is a presentation of S'.

Proof: If S' is sub-synchronizing of S then we can take $I_r = \bigcup_{i=1}^k I_i$ to be defined as in the above discussion. In that case Corollary 2.5 (c) shows that I_r has the property (i) and Proposition 3.3 shows that $G(I_r)$ is a presentation of S'. For (ii) we need to observe that every idempotent has a synchronizing word as a representative and by taking $e = [m]$, $s = [xm]$ and $t = [my]$ we have $st = [xmmy] = [xmy] \neq 0$.

Assume now that the syntactic monoid Syn has a right ideal satisfying the properties (i) and (ii). We will show that $G(I_r)$ is a presentation of a sub-synchronizing subshift S'. It is clear that every R-class of I_r corresponds to an irreducible component in $G(I_r)$ and vice versa. Let $e = [m]$ be an idempotent in I_r. Then $[m]m = [m]$ in $G(I_r)$. So $G(I_r)$ is a presentation of a sofic shift (i.e. every vertex in $G(I_r)$ has an edge coming in and an edge going out). m is synchronizing for $F(S)$: let $x, y \in F(S)$ such that $xm, my \in F(S)$; then $[xm] \leq_L [m]$ and $[my] \leq_R [m]$. By (ii) we get that $[xm][my] \neq 0$, i.e. $[xmmy] = [xmy] \neq 0$ and thus $xmy \in F(S)$. Thus m is magic for $F(S)$.

Let $M = \{m \in F(S) | \, [m] \text{ is an idempotent of } I_r\} \subset \mathcal{M}(S)$. Let S' be the sub-synchronizing of S defined via M. We will show that $G(I_r)$ is a presentation of S'. By the definition of S' and $G(I_r)$ it is clear that if $y \in F(S')$ then $mxy \in F(S)$ for some x and so y is in the right context of the vertex $[mx]$ in $G(I_r)$, i.e. $F(S') \subset L(G(I_r))$. If $y \in L(G(I_r))$ then there are vertices $[x_1]$ and $[x_2]$ such that $[x_1]y = [x_2]$. But there exists an idempotent $[m]$ in a maximal R-class of I_r such that $[mz] = [x_1]$, i.e. $[mzy] \neq 0$ and so $y \in F(S')$. ∎

COROLLARY 3.5: *Let $e \in \mathcal{I}_c$ be an idempotent and m be a magic synchronizing word such that $[m] = e$. Let I_e be the right ideal in* Syn *generated by e. Then $G(I_e)$ is the minimal synchronizing deterministic presentation of $\mathcal{S}(\mathbf{m})$.*

Proof: The proof follows from the fact that I_e satisfies the conditions (i) and (ii) from Proposition 3.4 and the proof of Proposition 3.4. Since m is magic synchronizing, it is easy to see that if $[mx]$ and $[my]$ have same right contexts in $G(I_e)$ then $[mx] = [my]$. So $G^r(I_e) = G(I_e)$, i.e. $G(I_e)$ is the minimal synchronizing deterministic presentation of $\mathcal{S}(\mathbf{m})$ (by Propositions 0.3 (a) and 1.3). ∎

The above corollary together with Proposition 2.6 show that we can obtain a synchronizing deterministic presentation for every sub-synchronizing subshift of S from right ideals of Syn in $\mathcal{I}_c$. If $S' = \mathcal{S}(\mathbf{M})$ is a sub-synchronizing subshift of S then $S' = \bigcup_{\mathbf{m}\in\mathbf{M}} \mathcal{S}(\mathbf{m})$. In this case, a synchronizing deterministic presentation for S' is $G(\bigcup_{e\in E} I_e)$ where $E = \{[m] | \exists \mathbf{m} \in \mathbf{M}, m^i \in \mathbf{m} \text{ for all } i = 1, 2, 3, \ldots\}$.

We need to point out that the regular D classes are in one-to-one correspondence with the equivalence classes in $\mathcal{E}$, but they don't preserve the natural order. It follows from Proposition 2.4 that if $D_{[m_1]} \leq D_{[m_2]}$ for two magic synchronizing words m_1, m_2, then $D_{[m_1]} = D_{[m_2]}$. So the order $\mathbf{m}_1 \leq \mathbf{m}_2$ in $\mathcal{E}$ does not transfer onto the corresponding order between the D classes.

A characterization of Syn if S is a synchronizing sofic shift was given in [7]. Proposition 3.4 allows us to give another characterization of synchronizing sofic shifts. Let $\mathcal{I}_c = \{[x] | x \text{ is a constant for } F(S)\}$.

COROLLARY 3.6: *The following four conditions are equivalent:*

(1) *S is a synchronizing sofic shift.*

(2) Syn *has a right ideal I such that*
   - (2.i) *for all non-zero $s \in$ Syn, $Is \neq 0$;*
   - (2.ii) *for all $s, t \in I$, if $st \neq 0$ then $st = t$.*

(3) Syn *has a right ideal I_r such that*
   - (3.i) *for all non-zero $s \in$ Syn, $I_r s \neq 0$;*
   - (3.ii) *for all $s, t \in$ Syn $-\{0\}$ and an idempotent $e \in I_r$, if $s \leq_L e$ and $t \leq_R e$ then $0 < st \leq_J e$;*
   - (3.iii) *an R-class of I_r is regular iff it is maximal in I_r.*

(4) *For all non-zero $s \in$ Syn, $\mathcal{I}_c s \neq 0$.*

Proof: The equivalence of (1) and (2) was shown in [7]. We need to show the equivalence of (1) and (3). But this follows from the fact that S is synchronizing iff it is sub-synchronizing of itself, by Proposition 3.4 and the observation that the condition (iii) is equivalent to: $G(I_r)$ is a presentation of S. The equivalence of (1) and (4) follows from the definition of $\mathcal{I}_c$, Corollary 2.5 and Proposition 0.2. ∎

We point out to the reader that condition (4) in the above corollary is essentially equivalent to Proposition 0.2. If we compare the properties (2.i), (3.i), and (4) we see that their only difference is in the ideals that they are referring to. The significance of (2) and (3) is that, in case of synchronizing sofic shift S, there are right ideals of Syn properly included in $\mathcal{I}_c$ with annihilator equal to 0.

In Corollary 3.6 we refer to two right ideals I and I_r. They are not necessarily equal. I that satisfies condition (2) also satisfies condition (3), but the converse is not necessarily true. So, a natural question is: how are I and I_r related to each other?

Let I be an ideal in Syn satisfying condition (2) of Corollary 3.6, and let I_r be an ideal in Syn satisfying condition (3) of Corollary 3.6. Let $\mathcal{I}$ be the union of D-classes that have a common element with I and let $\mathcal{I}_r$ be the union of D-classes that have a common element with I_r.

PROPOSITION 3.7: *$\mathcal{I}$ is an ideal and $\mathcal{I} \subset \mathcal{I}_r$. Moreover, if J is another right ideal which satisfies condition (2) of Corollary 3.6 and $\mathcal{J}$ is a union of all D-classes that have a common element with J, then $\mathcal{I} = \mathcal{J}$.*

Proof: We will show first that the maximal R-classes of I are regular. Let $[x]$ be an element in a maximal R-class. There is a $[z] \in I$ such that $[z][x] \neq 0$. So $[zx] = [x]$ by (2) (i), i.e. $[x] \leq_R [z]$. But $R_{[x]}$ is maximal, so $[x]R[z]$. Thus there is $[y]$ such that $[xy] = [z]$. Consider $[xy][xy] = [z][xy] = [zx][y] = [xy]$, i.e. $[z]$ is an idempotent in $R_{[x]}$ and thus $R_{[x]}$ is regular.

Next we show that maximal D-classes of $\mathcal{I}$ are also maximal D-classes of $\mathcal{I}_r$. Since $G(I_r)$ is a presentation of S, for each $x \in F(S)$ there is an idempotent $[n] \in I_r$ and a $y \in F(S)$ such that $[ny]x = [nyx]$ in $G(I_r)$. Let $[m]$ be an idempotent of I. Then there is an idempotent $[n]$ in I_r and $y \in F(S)$ such that $[nym] \in I_r - \{0\}$. But then there is a $[z] \in I$ (by (2) (ii)) such that $[z][nym] \neq 0$. Since I is a right ideal, $[zny] \in I$. But $[m] \in I$ and, by (2)(i), $[zny][m] = [znym] = [m]$, i.e. $[m] \leq_J [n]$. By Proposition 2.4, $D_{[m]} = D_{[n]}$.

Now we show that $\mathcal{I} \subset \mathcal{I}_r$. Let $[x] \in I$. There is an idempotent $[m] \in I$ such that $[m]x = [mx] \neq 0$. But then $[mx] = [x]$. There is an idempotent $[n]$ in I_r such that $[n]D[m]$, i.e. there are $z_1, z_2 \in F(S)$ such that $[z_1 n z_2] = [m]$. So $[z_1 n z_2 x] = [x]$, i.e. $[x] \leq_L [n z_2 x]$. Since I_r is a right ideal, $[n z_2 x] \in I_r$. But $D_{[n]} = D_{[m]}$ also implies that there are $z'_1, z'_2 \in F(S)$ such that $[z'_1 m z'_2] = [n]$ and so $[z_1z_1'mz_2'z_2x] = [x]$. Since I is a right ideal $[mz_2'z_2] \in I$, and so $[mz_2'z_2][x] = [x]$. But then $[z_1'][x] = [z_1'mz_2'z_2x] = [nz_2x]$, i.e. $[nz_2x] \leq_L [x]$. Hence, $[nz_2x]L[x]$ and $D_{[x]} = D_{[nz_2x]} \subset \mathcal{I}_r$.

$\mathcal{I}$ is an ideal: Let $[x],[y] \in$ Syn and $[n] \in \mathcal{I}$. If $[xny] = 0$ then $[xny] \in \mathcal{I}$. Assume that $[xny] \neq 0$. There is $[n'] \in I$ such that $[n]D[n']$ and so there are $z', z'' \in F(S)$ such that $[n] = [z'n'z'']$. There is an idempotent $[m] \in I$ and $x' \in F(S)$ such that $[mx'][xz'n'z''y] \neq 0$. Since I is a right ideal, $[mx'xz'] \in I$ and $[n'z''y] \in I$, i.e. $[mx'xz'n'z''y] = [n'z''y]$. So $[n'z''y] \leq_L [xz'n'z''y] = [xny]$ and $[xny] \leq_L [n'z''y]$, i.e. $[xny]D[n'z''y]$, and so $[xny] \in \mathcal{I}$.

Finally we see that $\mathcal{I} = \mathcal{J}$. Let $[y] \in \mathcal{J}$. There is an idempotent $[m] \in I$ and $z_1 \in F(S)$ such that $[mz_1][y] \neq 0$. So $[mz_1y] \leq_L [y]$. But J also satisfies the conditions (2) (i) and (ii), and hence there is an idempotent $[m'] \in J$ and $z_2 \in F(S)$ such that $[m'z_2mz_1y] \neq 0$. Since J is right ideal $[m'z_2mz_1] \in J$, and so $[m'z_2mz_1][y] = [y]$, i.e. $[y]L[mz_1y]$ and $[mz_1y] \in I$. Thus $D_{[y]} \subset \mathcal{I}$. By the symmetry of the argument we have that $\mathcal{I} = \mathcal{J}$. ∎

A right ideal I_r satisfying condition (3) is in general larger than any ideal satisfying condition (2). A right ideal I satisfying condition (2), by the Corollary 3.6, determines a unique ideal in Syn. That is the ideal that corresponds to the unique minimal synchronizing deterministic presentation of S ([7]).

We end our discussion with the following illustrations of Propositions 3.6 and 3.7.

Examples: 3.1. Consider the sofic shift presented in Figure 2. The syntactic monoid of its factor language is presented in Figure 5. (The idempotents are denoted by $*$ and the square brackets are omitted.)

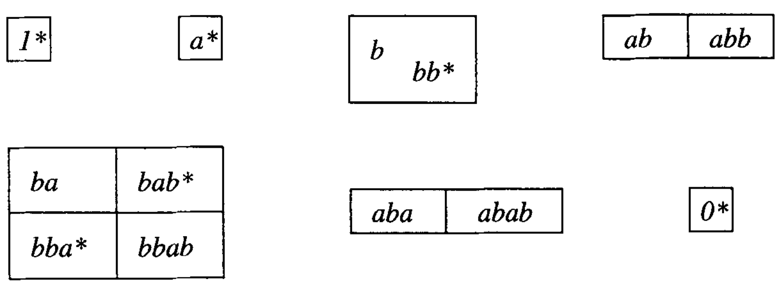

Figure 5. The syntactic semigroup of the factor language presented in Figure 2.

The ideal $\mathcal{I}_c$ contains the D-classes D_{ba}, D_{aba} and 0. The right ideal I_r (Proposition 3.4) is equal to $R_{ba} \cup 0$ or $R_{bba} \cup 0$. There is no right ideal I satisfying condition (2) in Corollary 3.7.

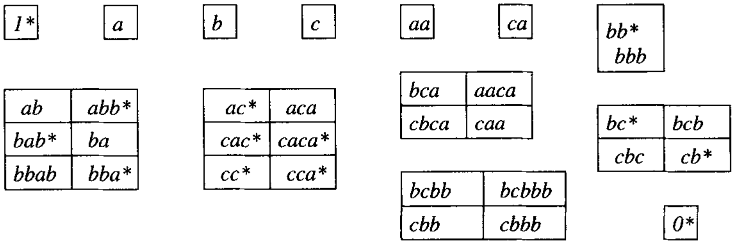

Figure 6. The syntactic semigroup of the factor language presented in Figure 3.

3.2. The sofic shift presented in Figure 3 (a) has a syntactic monoid presented in Figure 6. Since S is synchronizing, there is a right ideal $I = R_{bc} \cup R_{bca} \cup R_{bcbb} \cup 0$ satisfying condition (2) in Corollary 3.6 and is generated by the idempotent $e = [bc]$. Here $G(I)$ is the same presentation as in Figure 3 (a). The ideal generated by the constant words is $\mathcal{I}_c = \mathcal{I} \cup D_{ab} \cup D_{ac}$, where $\mathcal{I}$ is the ideal obtained by taking the union of D-classes intersecting I. For the right ideal I_r satisfying condition (3) in Corollary 3.7 we have few choices, like I, or $I \cup R_{ab}$, or $I \cup R_{ac}$. The sub-synchronizing subshifts S_1 and S_2 have presentations $G(R_{ab})$ and $G(R_{ac})$, respectively. For the third sub-synchronizing subshift $S_3 = S_1 \cup S_2$ we have a synchronizing determinisitic presentation $G(R_{ab} \cup R_{ac})$.

4. Discussion

To end this paper we would like to point out several observations about synchronizing sofic shifts. By reversing the definitions of deterministic and synchronizing, we can introduce notions of co-deterministic and co-synchronizing presentations.

We define a **co-deterministic presentation** of a sofic shift to be a presentation G such that the incoming edges at each vertex are labeled distinctly. Similarly, a **co-synchronizing** presentation is a presentation G with the following properties: for every vertex v there is a word m_v such that every path in G with label m_v starts at v. We say that S is co-synchronizing if it has a co-synchronizing presentation.

Clearly, the definitions of co-synchronizing and co-deterministic presentations are symmetric to the synchronizing and deterministic ones. Thus every property of synchronizing sofic shifts can be reversed for the co-synchronizing shifts (by reversing right Krieger cover to left Krieger cover, right ideals to left ideals, terminal subgraphs to initial subgraphs ...).

The definition of synchronizing words for a sofic shift S is symmetric and intrinsic. The same is true for the magic words. So, we have that the set of synchronizing (magic) words for S coincides with the set of co-synchronizing (magic) words. But there are synchronizing sofic shifts which are not co-synchronizing (see [7]).

An interesting case is when a sofic shift has a presentation which is both synchronizing and co-synchronizing. We call these presentations **bi-synchronizing.** It was shown in [9] that if an irreducible (or a shift of finite type) has a bi-synchronizing bi-deterministic presentation G_M, then this presentation is unique minimal in the following sense: Let G be any other irreducible presentation (in case of SFT, **any** other presentation). Then there is a 0-block map f from the graph shift S_G to S_{G_M} such that the following diagram commutes:

$$\begin{array}{ccc} S_G & \xrightarrow{f} & S_{G_M} \\ {\scriptstyle \Lambda_G}\big\downarrow & & \big\downarrow{\scriptstyle \Lambda_{G_M}} \\ S & = & S \end{array}$$

But there are irreducible SFT with no bi-synchronizing presentations ([7]).

It was shown in [7] that bi-synchronizing presentations are unique (up to a graph isomorphism) with a minimal number of vertices. An irreducible sofic shift is **almost finite type** (see [3]) if it has a unique minimal cover such that every other cover factors through it (as in the above diagram), not necessarily with a 0-block map. It would be interesting to see if bi-synchronizing sofic shifts (which are not necessarily irreducible) have the same property. The difficulty in this case arises from the fact that bi-synchronizing presentations are not necessarily deterministic nor co-deterministic (not even left or right closing, see [3]).

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